Assimilating Irregularly Spaced Sparsely Observed Turbulent Signals with Hierarchical Bayesian Reduced Stochastic Filters

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Motivation

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• In particular, we will use hierarchical Bayesian framework.

Standard Bayesian Approach

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Canonical Discrete-Time Filtering Problem:

$$u_{m+1} = f(u_m) + \sigma_{m+1}, \quad \sigma \sim \mathcal{N}(0, r),$$

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Solution: Apply the Bayesian Theorem:

$$P(U|\tilde{V}) \propto P(U)P(\tilde{V}|U).$$

Hierarchical Bayesian Approach

Consider $v \in V$ to be the random variable of interpolated observations at the regular model grid points. Our approach is to apply

$$\begin{aligned} P(U|\tilde{V}, V) &\propto P(U)P(V|\tilde{V}, U) \\ &\propto P(U)P(\tilde{V}|U, V)P(V|U). \end{aligned}$$

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Step 2: We apply $P(U)P(V|\tilde{V}, U)$ through a reduced stochastic Fourier based filter.

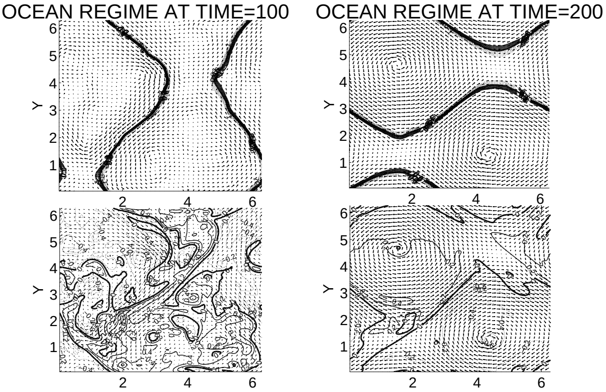
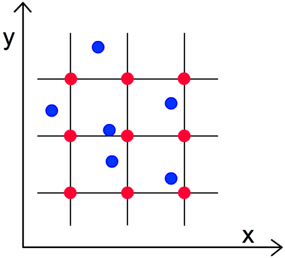


Figure: The 2 layer QG model with baroclinic instability, resolved with 128×128 grid points in a 2D periodic domain [Smith et al, 2002]. The radius of deformation is chosen to mimic ocean turbulence. The top panels show the barotropic velocity field (arrows) and streamfunction Ψ, (contour) and the bottom panels show the baroclinic velocity field and streamfunction Ψ (bottom) at two different times.

Given two-dimensional noisy, sparse observations from the solution to the two-layer quasi-geostrophic model with baroclinic instability, the first task is to interpolate to a regular 6×6 grid.



Kriging is a maximum likelihood estimator of a random field Z modeled by

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1. Estimate the mean μ using median polishing. [Cressie, 1993]
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We compare ordinary kriging with a deterministic linear interpolation.

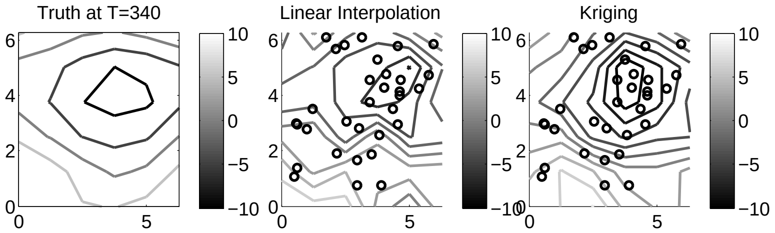

Figure: The true field (left) and the results of a linear interpolation (middle) and kriging interpolation (right).

Spatial Interpolation Results

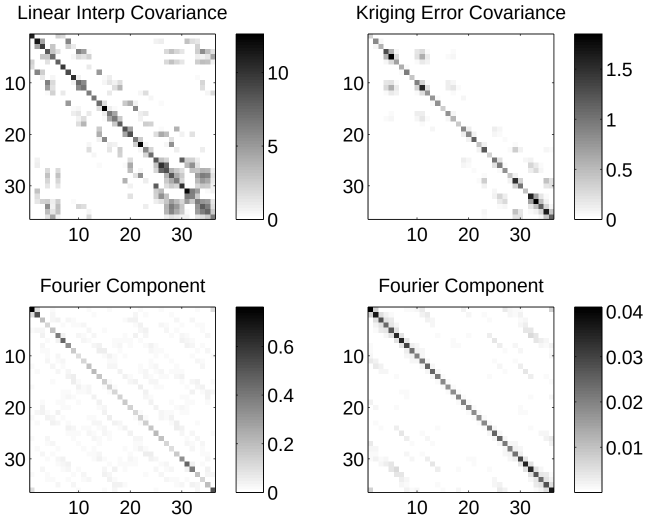

Figure: The noise covariance in physical space (top) and Fourier space (bottom).

Filtering: The Mean Stochastic Model

The next step is to apply a reduced stochastic Fourier based filter. The filter approximates the barotropic modes of the 2 layer QG model

$$\begin{aligned} \frac{\partial q}{\partial t} + J(\Psi, q) \quad &+ \quad \beta \frac{\partial \Psi}{\partial x} + \kappa \nabla^8 q \\ &+ \quad \left[J(\Psi^c, q^c) + U \frac{\partial \nabla^2 \Psi^c}{\partial x} - \kappa \nabla^2 \Psi^c \right] = 0 \end{aligned}$$

in Fourier space with

$$d\hat{\Psi}(t) = (-d + i\omega)\hat{\Psi}(t)dt + Fdt + \sigma dW(t).$$

[Madja and Harlim, Chapter 12, 2012]

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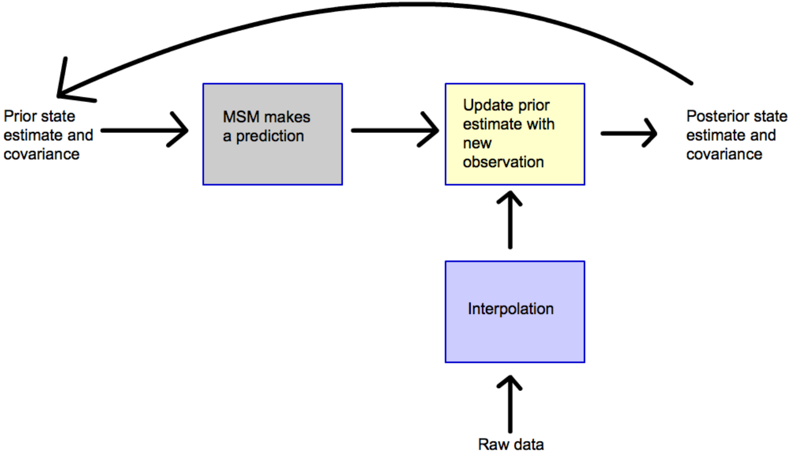
$\hat{\Psi}$: the horizontal Fourier component of the barotropic streamfunction Ψ,
$W(t)$: a complex-valued Wiener process,
d: damping,
ω: frequency,
F : constant external forcing,
σ: noise strength. [Madja and Harlim, Chapter 12, 2012]

The Kalman Filter

The Kalman Filter is a solution to these equations and produces estimates of the mean and covariance prior and posterior to observation



Filtering Results

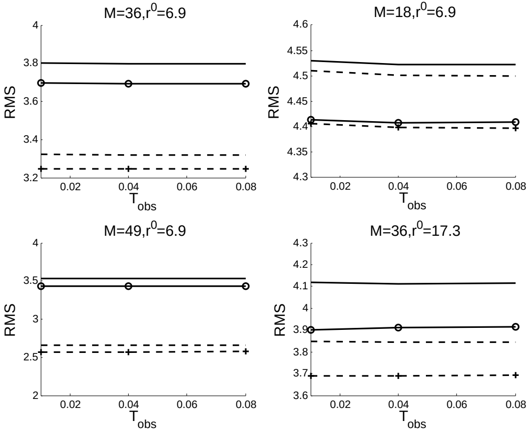


Figure: The RMS errors associated with each step: unfiltered kriging (dashes), filtered kriging (dashes with '+' sign), unfiltered linear interpolation (solid line), and filtered linear interpolation (solid line with circles).

Filtering Results: $M = 36$ and $r^0 = 17.3$

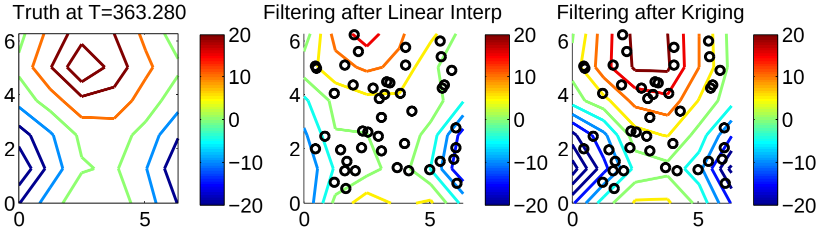


Figure: Filtering results at one particular time. The circles illustrate observation locations.

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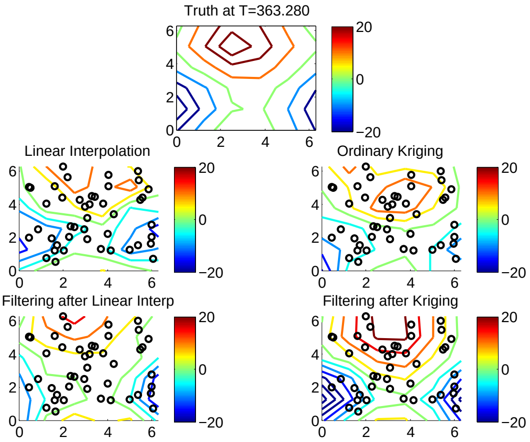

Figure: The true barotropic streamfunction (top), interpolated results (middle panels) and filtered results (bottom panels) at one particular time. The circles illustrate observation locations.

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- The Mean Stochastic Model is a very simple one, and we expect the results could be improved with other models.

References

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